# COVERING AND INDEPENDENT NUMBERS OF CERTAIN TYPES OF CAYLEY GRAPHS

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***ABSTRACT :*** *In this paper, we will study cayley graphs of certain types of groups. It is a representative graph for groups using a set of generators. And we will construct a general formula to compute covering and independence numbers of certain types of Cayley graphs.*

**Key words**: Graph, Group, Cayley graph, Covering numbers, Independence number.

## 1. INTRODUCTION

The relationship between graph theory and group theory is developed and improved exponentially. Using group theory as tools in graph theory and its applications.

In this part we introduce the basic definitions and concepts which will be used in the sequel.

Definition 1.1: A graph Γ = (V, E) consists of a finite nonempty set V=V (Γ) of n points together with a prescribed set E of q unordered pairs of distinct points of V.

We call V(Γ) the vertex-set of Γ , and E(Γ) the edge-set of Γ, often denoted by V and E respectively , the graph Γ will be called an (n,q)-graph where n is the number of vertices and q is the number of edges in Γ .

Each pair x= {u,v} of vertices in E (Γ) is an edge of Γ , and x joins u and v. Sometimes, we write x=uv, and say that u and v are adjacent vertices (by u adj .v), u and x are incident with each other, as are v and x.

Definition 1.2: The order of Γ is the number of vertices of Γ and denoted by | V (Γ) |.

Definition 1.3: A walk of graph Γ is an alternating sequence of points and lines $v_0\ x_1\ v_1\ x_2 \dots v_{n-1}\ x_n\ v_n$ (and sometimes $v_0\ v_1 \dots v_{n-1}\ v_n$), beginning and ending with points it is sometimes called a $v_0$-$v_n$ walk.

It is closed if $v_0 = v_n$ and is open otherwise. The number of edges occurring in a walk will be called the length of the walk.

Definition 1.4: A cycle is the walk with distinct n vertices and n≥3

(and thus necessarily all edges are distinct).

Definition 1.5: A path is an open walk with distinct vertices and edges. But if a walk with distinct edges is a trial.

Definition 1.6: A graph is K-regular if every vertex is connected to k other vertices through k-edges.

Definition 1.7: A subgraph of Γ is a graph having all of its vertices and edges in Γ . In other word a graph R is called a subgraph of a graph Γ if V (R) ≤ V (Γ) and E (R) ≤ E (Γ) .

Definition 1.8: A simple graph is undirected graph that has no loops and no more than one edge between any two different vertices.

Definition 1.9: A vertex and an edge are said to cover each other if they are incident.

Definition 1.10: If every vertex in graph Γ adjacent with 3 anther vertices then Γ is called cubic (trivalent graph).

Definition 1.11: Independent set of vertices is the set of all vertices in Γ in which every two vertices are not adjacent, and the maximum number of independent vertices denoted by $\beta_0(\Gamma)$.

Definition 1.12: Independent set of edges is the set of all edges in Γ in which every two edges are not adjacent, and the maximum number of independent edges denoted by $\beta_1(\Gamma)$.

Definition 1.13: two graphs $\Gamma_1$ and $\Gamma_2$ are said to be isomorphic (denoted by $\Gamma_1 \cong \Gamma_2$ ) if there exists a 1-1 correspondence between their vertex sets , which preserves adjacency .

Definition 1.14: A graph is connected if every pair of vertices there is at least the path joining them. A graph is that is not connected is called disconnected.

Definition 1.15: A digraph is strongly-connected, or strong, if every two vertices are mutually reachable.

Definition 1.16: A graph is complete graph $K_n$ if every pair of its vertices adjacent. Thus $K_n$ is regular of degree n-1.

Definition 1.17: the valency of vertex denoted by $val(v_i)$= number of edges incident to $v_i$ ( sometimes we called it degree of vertex $v_i$ and denoted by $d_i$ or deg $v_i$ ).

Definition 1.18: The distance $d_\Gamma(u,v)$ between two vertices u and v in Γ is the length of shortest path joining them if any ; otherwise $d_\Gamma(u,v)=\infty$ , if u=v then $d_\Gamma(u,v) = 0$ (in digraph the distance between two vertices u and v is length of any shortest such path).

Definition 1.19: The diameter d(Γ) of a connected graph Γ is the length of any longest geodesic (a shortest u-v path is called geodesic)

i.e d(Γ) = max {d (u,v) } ; u.v ϵ Γ.

Definition 1.20: A graph Γ is n-transitive, n≥1 if it has an n-rout and if there is always an automorphism of Γ sending each n-rout onto any other n-rout.

Definition 1.21: A group is an order pair (G,*) where G is a non-empty set and * is a binary associative operation on G which contains an identity (the natural element e) and inverse for each element.

Definition 1.22: If a subset H of a group G is itself a group under the operation of G, we say H is a subgroup of G.

Definition 1.23: Let G be a group and let $g_i$ ϵ G for i ϵ I if this {$g_i$ : i ϵ I } subgroup is all of G then {$g_i$ : i ϵ I} generates G and the $g_i$ are generates of G.

Definition 1.24: Let G be a group. A subset Ω of G is a generating set for group G if every element of G can be expressed as a product of elements of set Ω.

Definition 1.25: Let A be the finite set {1,2,…,n}. The group of all permutations of A is the symmetric group on n letters and is denoted by $S_n$.

Definition 1.26: A permutation of a set A is one to one function from A onto A.

Definition 1.27: An element of a group an involution if it has order 2 (i.e an involution is an element a such that $a \neq e$ and $a^2 = e$ where e is the identity element).

2. Group Representation by Cayley Digraphs.

In this part we will introduce the Cayley dighrap of group. Provides a method of visualizing the group and its properties.

A directed graph (or digraph) is a finite set of points called vertices, and a set of arrows called arcs (edges), connecting some of the vertices.

Let G be finite group G and Ω a set of generators for G.

We define a digraph Cay (Ω: G), called the Cayley digraph of G with generating set as follows.

1- Each element of G is a vertex of Cay (Ω:G)

2- For v and u in G , there is an arc(edge) from u to v if and only if

u = vx , for some x ϵ **Ω**

In Cayley digraph method we proposed that each generator by assigned a color , to know which particular generator connects two vertices , and that the arrow joining v to vx be colored with color assigned to x , we called the resulting figure the color graph of the group . Rather than use colors to distinguish the different generators, we will use solid arrows, dashed arrows, and dotted arrows.

In general, if there is an arc from v to u, there need not be an arc from u to v, note that there are several ways to draw the digraph of a group given by a particular generating set. However, it is not the appearance of the graph that is relevant but the manner in which the vertices are connected.

These connections are uniquely determined by the generating set. Thus, distances between vertices and angles by the arcs have no significance.

It is important to note that Cayley graph of the same group can vary depending on which set generates the group.

The following examples illustrate the representation of certain groups by Cayley digraphs. For example,

The Cayley digraph for the symmetric group $S_3$ with the generating set

**Ω** = {(1,2),(1,2,3)}

The incident function is constructed as follows

| | |
|---|---|
| (e)(1,2,3) = (1,2,3) | (1,2,3) (1,2) = (1,3) |
| (1,3,2)(1,2,3) = (e) | (1,3,2) (1,2) = (2,3) |
| (1,2,3)(1,2,3) = (1,3,2) | (e) (1,2) = (1,2) |
| (1,3) (1,2,3) = (1,2) | (1,2) (1,2) = (e) |
| (1,2) (1,2,3) = (2,3) | (1,3) (1,2) = (1,2,3) |
| (2,3) (1,2,3) = (1,3) | (2,3) (1,2) = (1,3,2) |

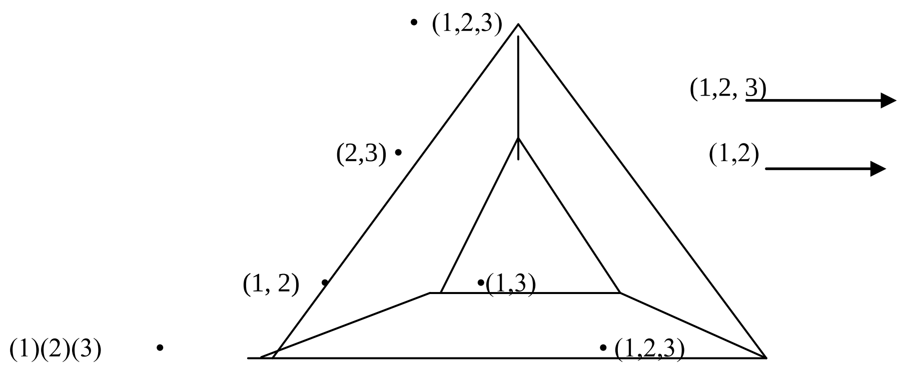



**Figure 2.1**

**Cay ({(1,2),(1,2,3)} : $S_3$)**

From this a Cayley digraph for the symmetric group $S_3$ with the generating set **Ω** = { (1,2) , (1,2,3) } . by looking at the identity element e , we can deduce that the solid arrow represents multiplication by (1,2,3) , because starting at the identity e , and following the solid arrow yields (1,2,3) . By the same logic, the dashed arrow represent (1,2) . The element (1,2,3) is of order 3 because starting at the identity and following the solid arrow once yields (1,2,3) , follow it again and you get to (1,3,2) , follow it once more and you get back to e, therefore applying the solid arrow three times is equiregular to the identity . By the principal, (1,2) had order 2 . a quick way to see that an element in the generating set order 2 is to look and see if it has a double-headed arrow , i.e. an arrow on both sides of the arc.

The Cayley digraph illustrates several interesting facts about $S_3$.

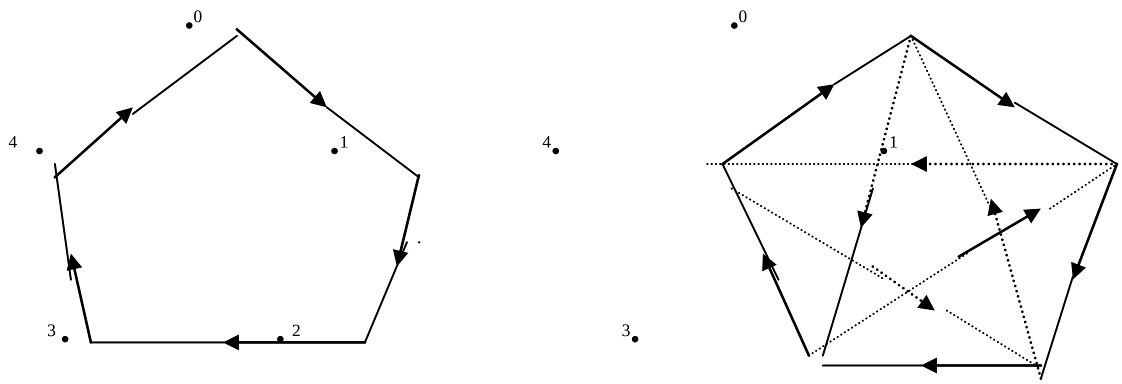
The Cayley digraph shows us that $S_3$ is non-commutative group. this can be seen by starting at any element , say (1,3) and following the solid arrow and then the dashed arrow , which yields e , then start at (1,3) again and follow the dashed arrow and then the solid arrow , this results in (1,3,2). Since e is different that (1, 3, 2) then $S_3$ is non-commutative group .

The Multiplication table of the group can be recovered from the Cayley digraph. As previously stated , (1,2,3) corresponds to traveling the solid arrow , therefore let (1,2,3) = S , and by the same logic , let (1,2) = D .

Then using this SD notation the rest of the elements can be represented in the same manner. (1,3) = SD , (1,3,2) = SS , and (2,3) =DS . Using this notation the multiplication table can be recovered by starting at the identity and traveling the corresponding arrows.

For example (1,3,2)(1,3) = SSSD = (1,2) and (2,3)(1,3) = DSSD = (1,2,3).

For example the Cayley digraph for the cyclic group $Z_5$ with generating set **Ω** = {1} ($Z_5$=<1>)




Cay ({1}: $Z_5$) Cay({1,3}:$Z_5$)

**Figure 2.3: Cay ({1}:$Z_5$) and Cay ({1,3}:$Z_5$)**

On the left figure the Cayley digraph of $Z_5$ with generating set {1} .

In the Cayley digraph each of the elements of the group $Z_5$ are the vertices of the digraph . The solid arrow represents addition by 1 , which is the only element in the generating set .The Cayley digraph illustrates several things about $Z_5$ with generating set {1}. The first point of interest is that 1 is order 5 , because if you start at the identity 0, and add 1 five times which equivalent to following the solid arrow five times , you get back to the identity . Another property that should be noted is $Z_5$ is cyclic because there is only one kind of arrows , which implies that there exist a generating set with only one element . (Willis, S [6])

Note : In general the Cayley digraph for $Z_n$ = <1> is $C_n$.

We can see more interesting examples for Cayley digraph in (Gallian.J.[2]) (Gross.J.[3]).

Now we will introduce some important theorems related to Cayley graph of groups contain the basic constructive properties.

An arbitrary graph Γ is said to be a Cayley graph if there exist a group G and a generating set Ω such that Γ is isomorphic to the Cayley graph for G and Ω .

Theorem 2.1

The complete graph $K_{2n+1}$ is a Cayley graph for group $Z_{2n+1\,+}$ with generating set {1,2,…,n} .

Proof

Let $\Gamma=K_{2n+1}$ is complete graph that's mean every vertex $v_i$ in V (Γ)

Must adjacent with 2n vertices of Γ . Now when draw Cayley graph for group $Z_{2n=1}$ with generating set {1,2,…,n} . WLOS first ny generating 1 we draw the cycle $C_{2n+1}$ from $v_0$ and traverse all the vertices in closed path to $v_{2n+1} = v_0$ (since in $Z_{2n+1}$ , 0 equal 2n+1)

(i.e $v_0$ adj . $v_1$ and $v_0$ adj. $v_{2n}$)

Now by generating 2 we draw the all edges in the following form

$v_0$ adj. $v_2$ $v_1$ adj. $v_3$

$v_2$ adj. $v_4$ $v_3$ adj. $v_6$

$v_{2n}$ adj. $v_1$ $v_{2n-1}$ adj. $v_0$

Following the same argument , every vertex of Γ adjacent with all vertices of Γ . Thus , the constructed graph is a complete graph $K_{2n+1}$ .

For example, the complete graph $K_7$ is a Cayley graph for group $Z_7$ with generating set {1,2,3}. Figure 2.2 illustrates this for $K_7$

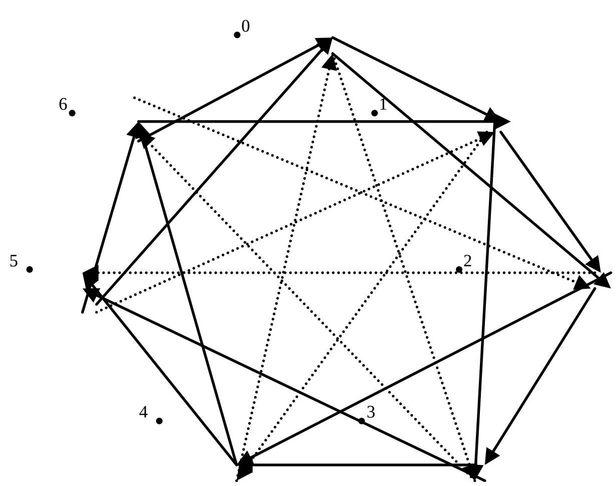



**Figure 2.2**
**Cay ({1,2,3}:$Z_7$)**

Note: the complete $K_{2n}$ is a Cayley graph for group $Z_{2n}$ with generating set {1,2,…,n} such that appear bidirected –arc with generator n .

A graph Γ is vertex-transitive if for all vertex pairs u,v ϵ V (Γ) , there is an automorphism of Γ that maps u to v .

(i.e. if Φ: Γ  Γ ; automorphism if u adjacent to v , then Φ (u) adjacent to v)

Lemma 2.1 (Biggs.N.[1]):

For any group G , the Cayley digraph is vertex-transitive .

(i.e every Cayley digraph is vertex-transitive).

For inctent the Petersen graph is smallest vertex-transitive but is not a Cayley graph, since its automorphism group has no transitive subgroup of order 10.

Lemma 2.2 (Ruskey.F.[5]).

Every Cayley digraph is strongly-connected.

Theorem 2.2 (Sabidussi.G.[7]).

Every vertex-transitive graph is homomorphic image of a Cayley graph.

An element of a generating set Ω of order 2 called an involution would cause doubled edges to appear in the Cayley graph. Collapsing such doubled edges to a single edge enhances the usefulness of Cayley graphs in algebraic specification. This new graph known Cayley graph is the underlying graph of the Cayley digraph (Cayley graph).

For example, let G be the symmetric group $S_3$ G = {e, (1,2),(1,3),(2,3),(1,2,3),(1,3,2)}.

Let the set of generators Ω consists involutions Ω = {(1,2),(1,3),(2,3)}

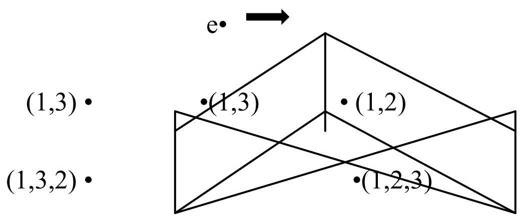



**Figure 2.3**
**Cay ({(1,2),(1,3),(2,3)}: $S_3$).**

3. Covering and Independent Nubers of Cayley graphs.

In this part we will introduce covering number and independent number of Cayley graph of a group.

Theorem 3.1 (Ruskey.F.[5]).

A set Ω = {$t_1$,$t_2$,…,$t_k$} of transpositions generates $S_n$ if and only if undirected graph associated with Ω (denoted by G (Ω) with n vertices where each edge denotes one of transpositions) is connected.

For example,

G(Ω) with 4 vertices (associated with $S_4$) there two forms either

1- Line graph (p4)

2- Star graph ($k_{1,3}$)

That's mean $S_4$ generated by the set of transpositions

1- Ω = {(1,2),(2,3),(3,4)}

2- Ω ={(1,2)(1,3),(1,4)} (this graph we can see in [5]).

A vertex and edge are said to be cover each other are incident, a vertex cover set for a graph Γ is a set of vertices which cover all the edges of a graph Γ and the smallest number of vertices that cover all edges of the graph called vertex covering number denoted by $\alpha_0(\Gamma)$ usually $\alpha_0$, while the edge cover set of Γ is a set of edges that cover all the vertices of Γ and the smallest number such number denoted by $\alpha_1(\Gamma)$ usually $\alpha_1$ is the edge covering number.

Note: every graph consists of tree and tree cover. (i.e $\Gamma = T \cup T^c$).

A tree is a connected graph with no cycles, i.e spanning tree and the tree cover denoted by $T^c$ is the set of edges which is cover the rest of edge of T.

The eccentricity e (v) of a vertex v in a tree T is max d (u , v) for all u in T, note that the maximum eccentricity is the diameter and the radius r(T) is the minimum eccentricity of the vertices of T.

Theorem 3.2: let Γ be a Cayley graph of a group G with k generators and radius r,2r = d(Γ), where d (Γ) is a diameter of Γ. Then

(i) $\alpha_0(\Gamma) = 1+ \sum_{i=0}^{n} k\,(k-1)^{2i-1}$ ; $n = \frac{r-2}{2}$ , if r is even integer

(ii) $\alpha_0(\Gamma) = \sum_{i=0}^{n} k\,(k-1)^{2i}$ ; $n = \frac{r-1}{2}$ , if r is odd integer

Proof.

Let Γ be a Cayley graph of a group G with k generators, so Γ is

A k-regular graph and let radius of Γ be r, so the tree of Γ has r levels

$L_0, L_1, \ldots, L_r$ and there is one vertex in $L_0$ and k vertices in $L_1$ and k(k-1) in $L_2$ and $k\,(k-1)^{r-1}$ vertices in $L_r$.

Let r be an odd integer, to scan the minimum number of vertices which cover the edge of $T^c$, so we have $k(k-1)^{r-1}$ vertices and in level $L_{r-2}$ we have $k(k-1)^{r-3}$ vertices i.e. choose off and on level following the same argument if r is odd we have

$K\,(k-1)^{r-1} + k\,(k-1)^{r-3} + \ldots + k(k-1)^4 + k\,(k-1)^2 + k$ , so

$\alpha_0\,(\Gamma) = \sum_{i=0}^{n} k\,(k-1)^{2i}$ ; $n = \frac{r-1}{2}$

Now, if r is even, the minimum number of vertices, will be

$K\,(k-1)^{r-1} + k\,(k-1)^{r-3} + \ldots + k\,(k-1)^3 + k\,(k-1) + 1$ , so

$\alpha_0\,(\Gamma) = 1 + \sum_{i=0}^{n} k\,(k-1)^{2i+1}$ ; $n = \frac{r-2}{2}$ .

Theorem 3.3: let Γ be a Cayley graph of a group G with k generators and radius r, 2r = d (Γ). Then edge cover number $\alpha_1$ (Γ):

$\alpha_1\,(\Gamma) = \sum_{i=0}^{n} k\,(k-1)^{2i}$ ; $n = \frac{r-1}{2}$ , if r is odd, and

$\alpha_1(\Gamma) = 1 + \sum_{i=0}^{n} k\,(k-1)^{2i+1}$ ; $n = \frac{r-2}{2}$ , if r is even.

Proof: Let Γ be a Cayley graph of a group G is a k-regular and let radius r of Γ equal half the diameter of Γ.

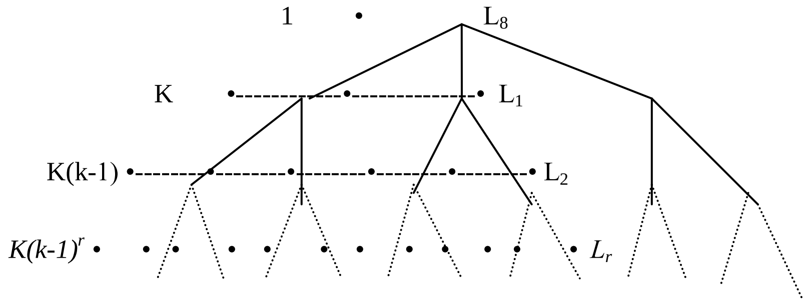



**Figure 3.1**

Now to cover vertices in last level $L_r$ we need $k\,(k-1)^{r-1}$ edges and the above set of vertices in level $L_{r-2}$ and $L_{r-3}$ we need $k\,(k-1)^{r-3}$ edges.

Following the same argument and if r is odd the minimum number of edge which cover all vertices of the tree and tree cover of the graph will be $\sum_{i=0}^{n} k\,(k-1)^{2i}$ ; $n = \frac{r-1}{2}$ ,

if r is odd

Similarly if r is even we need $1 + \sum_{i=0}^{n} k\,(k-1)^{2i+1}$ ;

$$n = \frac{r-2}{2}$$

since we need an edge to cover the root, thus the result holds.

For example,

to compute vertex (edge) cover number for following graphs:

r = 2 , d = 4 , 3-regular

$\alpha_1(\Gamma) = 6+1 = 7$

$\alpha_0(\Gamma) = 6+1 = 7$

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